Coupled wave theory

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ECE 299 Holography and Coherence Imaging Lecture 7

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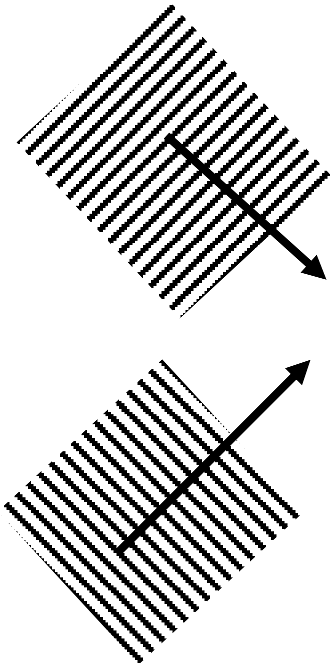
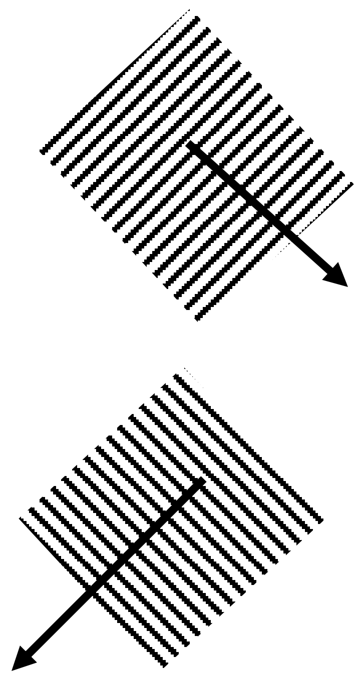
What is coupled wave theory?



Interacting medium (e.g. holographic emulsion)

Two or more waves interact in a medium (e.g. holographic emulsion) altering each other.

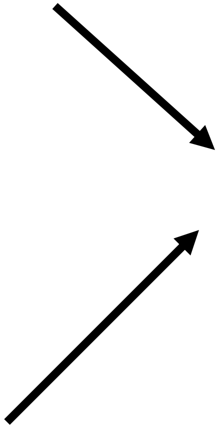


Interacting medium (e.g. holographic emulsion)

In holography, waves are coupled by a pattern recorded in the emulsion.

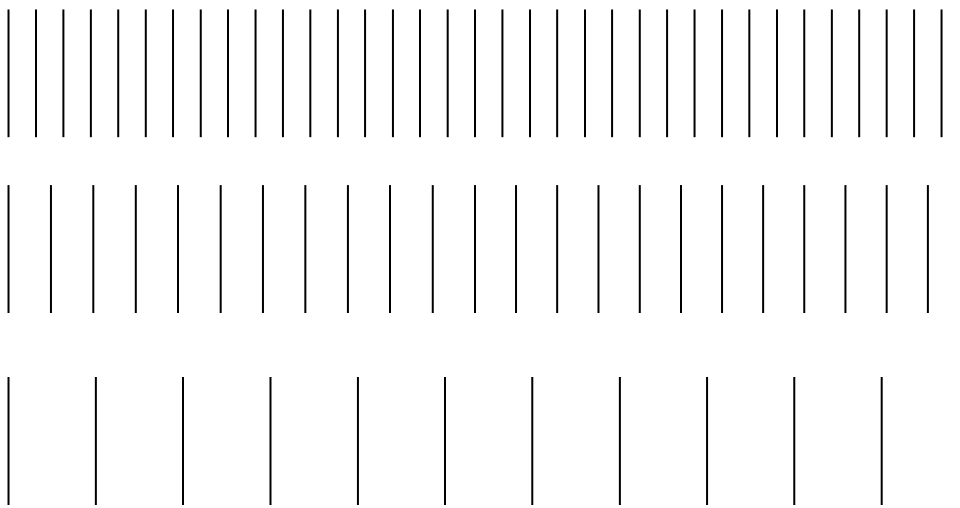
Volume holography

Two plane waves interfere inside a medium.



$$I(x) = \left| E_0 \exp(ik_1 x) + E_0 \exp(ik_2 x) \right|^2$$

Intensity of two superimposed plane waves



Wave #1 — Spatial frequency $k_1/2\pi$

Wave #2 — Spatial frequency $k_2/2\pi$

Interference pattern — Spatial frequency $(k_2 - k_1)/2\pi$

The hologram records a periodic pattern which has a spatial frequency given by the difference between the spatial frequencies of the interacting waves.

The pattern recorded by two plane waves (consider x direction only)

$$I(x) = \left| E_0 \exp(ik_1 x) + E_0 \exp(ik_2 x) \right|^2$$

$$I(x) = 2E_0{}^2 + 2\,\mathrm{Re}\left\{ E_0{}^2 \exp(ik_1 x) \exp(-ik_2 x) \right\}$$

$$I(x) = 2E_0{}^2 + 2\,\mathrm{Re}\left\{ E_0{}^2 \exp\left[i(k_1 - k_2)x \right] \right\}$$

$$I(x) = 2E_0{}^2 + 2E_0{}^2 \cos\left[(k_1 - k_2)x \right]$$

This is the periodic pattern recorded in the emulsion.

The emulsion electric permittivity changes in proportion to the intensity dose on the film.

$$\varepsilon(x) = \varepsilon_m + \alpha I(x) = \varepsilon + \Delta\varepsilon \cos\left[(k_1 - k_2)x \right]$$

There are many mechanisms for this photosensitivity (photochemical change, trapped charge, etc.)

The wave equation in a periodic hologram

$$\nabla^2 U + \mu\omega^2 \varepsilon(x) U = 0$$

Wave equation in inhomogeneous medium (scalar approximation).

$$\nabla^2 U + \mu\omega^2 \left(\varepsilon + \Delta\varepsilon \cos\left[(k_1 - k_2)x\right]\right) U = 0$$

Wave equation in periodically modulated permittivity medium

We use coupled wave theory to approximately solve this equation for two incoming plane waves.

Assumption of this derivation: the incoming plane waves vary spatially on a length scale much bigger than a wavelength (slowly varying envelope approximation).

Coupled wave theory.

$$U(x,z) = R(z)\exp(ik_{rx}x + ik_{rz}z) + S(z)\exp(ik_{sx}x + ik_{sz}z)$$

Express the field *U(x,z)* as a sum of two slowly varying plane waves R & S.

$R(z)$ and $S(z)$

Slowly varying amplitudes in z direction of R & S waves.

$$\exp(ik_{rx}x + ik_{rz}z) \qquad \exp(ik_{sx}x + ik_{sz}z)$$

X component of spatial frequency of plane waves

X component of spatial frequency of plane waves

Coupled wave theory (contd).

Insert $U(x,z)$ into the wave equation…

…expand out all of the derivatives…

…omit of the terms proportional to $\frac{d^2 R}{dz^2}$ and $\frac{d^2 S}{dz^2}$

because $R(z)$ & $S(z)$ are slowly varying.

Some terms are proportional to $\exp(ikr_z z)$ and some are proportional to $\exp(iks_z z)$. We separate these into two equations because the spatial oscillations at these frequencies are "out of phase" and interact very little.

Yet more coupled wave theory

We also remove a common propagation phase $\exp[ik(k_{rz}+k_{sz})z/2]$ and we get the two coupled differential equations:

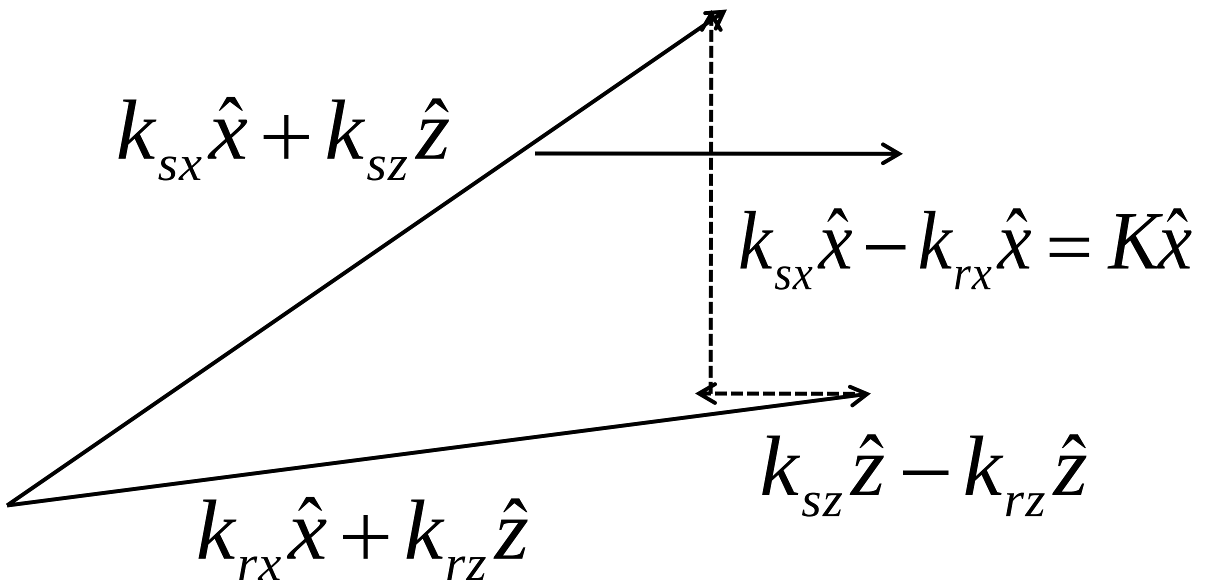
$$ik_{rz}\frac{dR}{dz}+\frac{k^2}{2}\frac{\Delta\varepsilon}{\varepsilon}S(z)\exp\left[i(k_{sz}-k_{rz})z\right]=0$$

$$ik_{sz}\frac{dS}{dz}+\frac{k^2}{2}\frac{\Delta\varepsilon}{\varepsilon}R(z)\exp\left[i(k_{rz}-k_{sz})z\right]=0$$

Note k_{rx}-$k_{sx}=K$ to make the x plane wave components cancel the hologram phase.

$$k^2=\mu\varepsilon\omega^2 \qquad k_{sz}{}^2+(k_{rz}-K)^2=k^2$$

Phase matching condition



$$k_{sz}^{\ 2}+(k_{rz}-K)^2=k^2$$

How to solve these equations

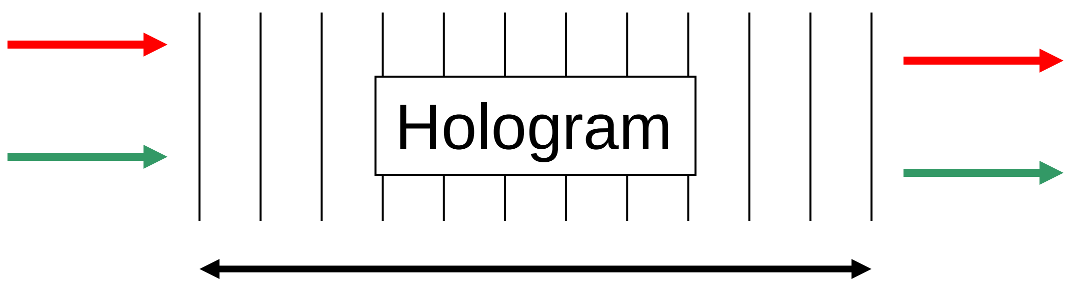
Define $\tilde{S}(z) = S(z)\exp\left[i(k_{sz} - k_{rz})z\right]$ and substitute…

$$ik_{rz}\frac{dR}{dz} + \frac{k^2}{2}\frac{\Delta\varepsilon}{\varepsilon}\tilde{S}(z) = 0$$

$$ik_{rz}\frac{d\tilde{S}}{dz} - k_{sz}(k_{rz} - k_{sz})\tilde{S} + \frac{k^2}{2}\frac{\Delta\varepsilon}{\varepsilon}R(z)\exp\left[i(k_{rz} - k_{sz})z\right] = 0$$

$R(0) = R_0$

$\tilde{S}(0) = 0$



$R(z)$?

$\tilde{S}(z)$?

$z = 0$ $z = d$

Initial conditions, no reference, a signal wave

How to solve these equations (contd.)

Use guess of sum of complex exponentials and solve the indicial equation. Back substitute and you find

Insert boundary conditions, solve for constants of integration, and you get…

$$R(z) = R_0 \exp\left(\frac{i\Delta k_z z}{2}\right)\left[\cos(\gamma z) - \frac{i\Delta k_z}{2\gamma}\sin(\gamma z)\right]$$

$$S(z) = \frac{iR_0 k^2}{2k_{sz}\gamma}\frac{\Delta\varepsilon}{\varepsilon}\exp\left(\frac{-i\Delta k_z z}{2}\right)\sin(\gamma z)$$

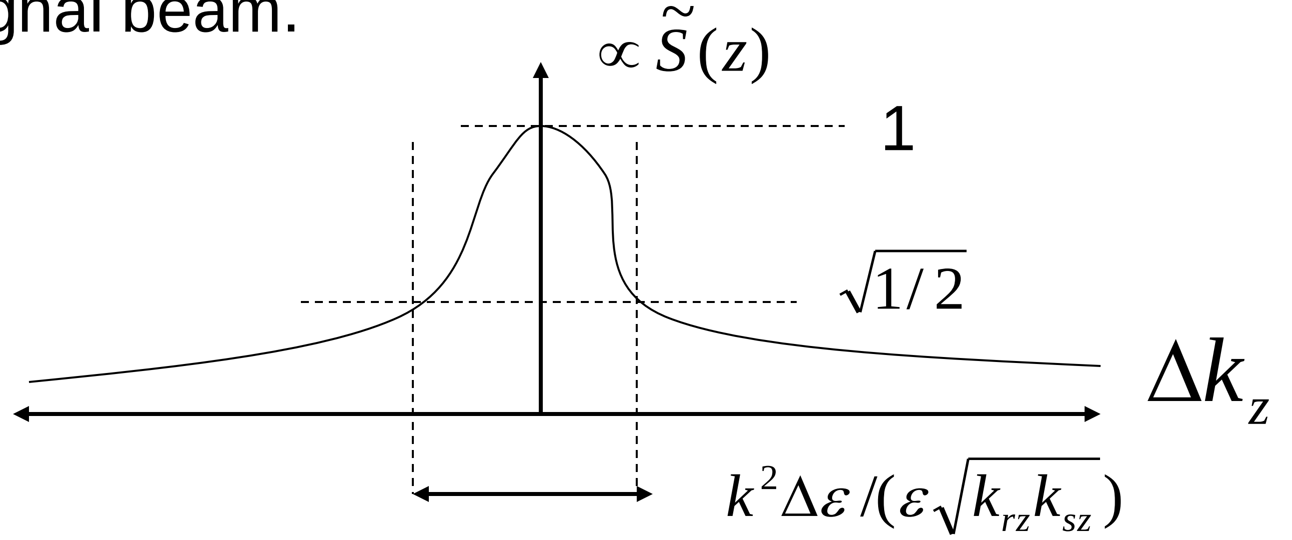
$$\Delta k = k_{rz} - k_{sz} \qquad \gamma = \frac{1}{2}\sqrt{\Delta k_z^2 + k^4 \Delta\varepsilon^2 /(k_{rz}k_{sz}\varepsilon^2)}$$

So what does this solution mean?

S(z) is proportional to γ^{-1}

$$\tilde{S}(z) \propto \left(\Delta k_z^2 + k^4 \Delta\varepsilon^2 /(k_{rz} k_{sz} \varepsilon^2)\right)^{-1/2}$$

The larger Δk_z, the more phase mismatch and the less power exchanged from the reference to the signal beam.



Diffraction efficiency

For the ideal case $\Delta k_z = 0$ with no phase mismatch, we find

$$R(z) = R_0 \cos(\gamma z) \qquad S(z) = iR_0 \sin(\gamma z)$$

Maximum of power to the signal occurs when $sin(\gamma z)=0$ or $\gamma z = \pi/2$

Diffraction efficiency is 1 when $\frac{k_{rz}}{k} = \frac{k_{sz}}{k} = \frac{\pi \Delta\varepsilon}{\varepsilon\gamma\lambda}$

Efficiency at power transfer η (diffraction efficiency)

$$\eta = \left| \frac{S}{R_0} \right|^2$$

And now for the simulations….

I performed a simulation of coupled wave theory.

Instead of these equations which are approximate but analytically tractable, I used the Finite Difference Time Domain (FDTD) method.

Simulated forming a hologram, reconstructing a hologram, Bragg diffraction.