we need to understand the processes of propagation and measurement of the laser field. These topics are considered in the next three sections.

1.3 SCATTERING

The goal of optical sensing is to discover properties of a remote object from measurements of the optical field. Solution of this problem requires an understanding of three transformations:

1. the transformation between properties of the remote object and the optical field in the vicinity of the object,
2. the transformation between the field at the object and the field at the sensor and
3. the transformation between the field at the sensor and measured data.

The main goal of optical sensor design is to encode the relationship between the field and object and between the field and measurements such that targeted object features can be determined. Holography is a particular solution to this problem.

The design of transformation (3) is the focus of much of this text. Prior to beginning this discussion, we briefly consider transformations (1) and (2) in this section and section 1.4. While one may explore diverse relationships between the object and the field, such as two photon generation and flourescence, holographic systems rely on laser illumination. Transformation (1) consists of *scattering* of the illumination by the object. Transformation (2), propagation of the field from the object to the sensor occurs via the process of *diffraction*. Fig. 1.3 illustrates the basic geometry of scattering and diffraction for a 3D object illuminated by a plane wave. Our goal is to find the transformation from field in the object space indexed by the xyz coordinates to the field in the diffraction space indexed by $x'y'z'$.

We assume that the illumination consists of the plane wave $Ae^{i2\pi\mathbf{u}_o\cdot\mathbf{r}}$, where we consider a *scalar wave* for simplicity. In contrast with the $\mathbf{E}$ field, the amplitude of the scalar wave is not a vector. The scalar wave may be considered as the amplitude of a single polarization component of the $\mathbf{E}$ field. We find through most of the text that analysis of wave propagation and recording using scalar quantities provides sufficient understanding of holographic systems. The student should be aware, however, that detailed design should account for polarization effects.

The object is characterized by magnetic permeability $\mu(\mathbf{r})$ and electric permittivity $\epsilon(\mathbf{r})$. In general, both μ and ϵ may be complex and tensor valued. For simplicity we assume that the object is not magnetic (e.g. $\mu(\mathbf{r}) = \mu_o$. The permittivity may be separated into a vacuum component and a component due to the polarizability such that

$$\epsilon(\mathbf{r}) = \epsilon_o(1 + \chi(\mathbf{r})), \tag{1.21}$$

where $\chi(\mathbf{r})$ is the electric susceptibility of the object.

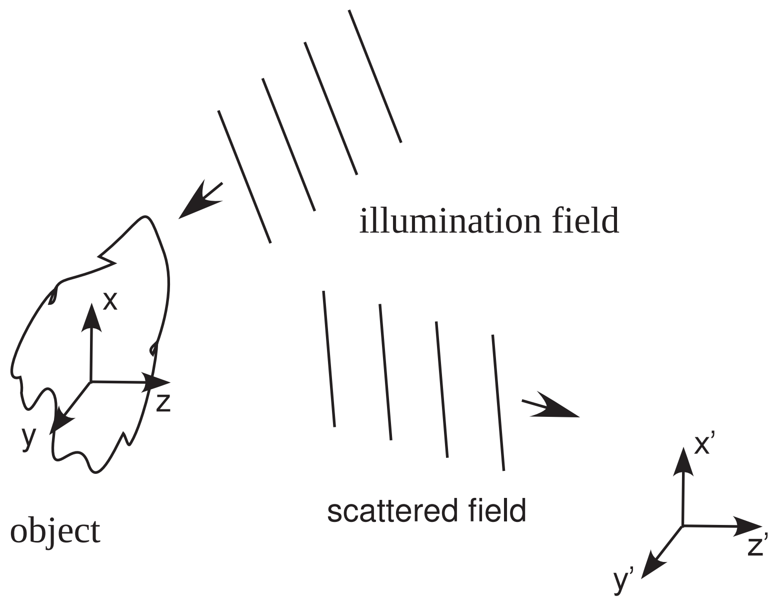

Figure 1.3. Scattering geometry for diffraction.

Since we are interested only monochromatic fields, we may reduce Eqn. 1.7 to the *Helmholtz wave equation*

$$\nabla^2\psi + 4\pi^2\mu\epsilon\left(1 + \chi(\mathbf{r})\right)\nu^2\psi = 0 \tag{1.22}$$

This equation may be expressed as the simple wave equation with a source term added due to scattering, e.g.

$$\left(\nabla^2 + k^2\right)\psi(\mathbf{r}) = k^2\chi(\mathbf{r})\psi(\mathbf{r}) \tag{1.23}$$

where $k = 2\pi\nu\sqrt{\mu\epsilon}$.

Eqn. (1.23) may be transformed into an integral equation using the *Green function* for the 3D scalar wave equation [3]. The Green function solves the wave equation with an impulse source term

$$\left(\nabla^2 + k^2\right) G\left(\mathbf{r}, \mathbf{r}'\right) = -4\pi\delta\left(\mathbf{r} - \mathbf{r}'\right) \tag{1.24}$$

The Green function for radiation from the object in unbounded homogenous space is

$$G\left(\mathbf{r}, \mathbf{r}'\right) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{1.25}$$

Using the Green function, the reader may show as an exercise (problem 1.6), that

$$\psi(\mathbf{r}) = \psi_o(\mathbf{r}) - \frac{k^2}{4\pi}\int\int\int \chi(\mathbf{r}')\psi(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d\mathbf{r}' \tag{1.26}$$

where $\psi_o(\mathbf{r})$ is a solution of the homogenous Helmholtz equation, $d\mathbf{r}'$ indicates the volume differential and the integral is over the volume of the scattering object. $\psi_o(\mathbf{r})$ is the field that would be present in the absence of the object (e.g. if $\chi(\mathbf{r}) = 0$), which in this case is the illumination field.

One approach to solving Eqn. (1.26) relies on iterative substitution. By substituting the right hand side of the equation for the field under the integrand, one obtains

$$\begin{aligned}\psi(\mathbf{r}) = \psi_o(\mathbf{r}) \quad & -\tfrac{k^2}{4\pi}\int\int\int \chi(\mathbf{r}')\psi_o(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d\mathbf{r}' \\ & +\tfrac{k^4}{16\pi^2}\int\int\int\int\int\int \chi(\mathbf{r}'')\chi(\mathbf{r}')\psi(\mathbf{r}'')G(\mathbf{r},\mathbf{r}'')G(\mathbf{r},\mathbf{r}')d\mathbf{r}'d\mathbf{r}''\end{aligned} \tag{1.27}$$

This recursive solution is called a “Born expansion.” The utility of the expansion comes from the approximation that the scattered field is much weaker than the illumination field. The expansion may be regarded as a power series in the scattering suceptibility. Under the assumption that $\chi(\mathbf{r}) \ll 1$, one may assume that higher order terms produce less scattering than the first order term. Truncation of this series at just the first order is called the “Born approximation.” Under the Born approximation, the scattered field is

$$\psi_s(\mathbf{r}) = -\frac{k^2}{4\pi}\int\int\int \chi(\mathbf{r}')\psi_o(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d\mathbf{r}' \tag{1.28}$$

It is interesting to note that the scattered field under the Born approximation is the convolution of the susceptibility-field product with the Green function. The Green function is the “impulse response” for the scattering process. Since the transformation is a convolution, one may apply the Fourier convolution theorem to express the Fourier transformation of the scattered field. We make extensive use of Fourier analysis in considering optical fields in this text, using the conventions and notations of [1]. In particular, we write the Fourier transform of $f(x)$ as

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iux}dx \tag{1.29}$$

Applying the 3D Fourier transform to Eqn. (1.30) yields

$$\hat{\psi}_s(\mathbf{u}) = -\frac{k^2 A}{4\pi} \frac{\hat{\chi}(\mathbf{u} - \mathbf{u}_o)}{\lambda^2 |\mathbf{u}|^2 - 1} \tag{1.30}$$

where we apply the Fourier shift theorem on to note that with plane wave illumination the Fourier transform of $\chi(\mathbf{r})\psi_o(\mathbf{r})$ is $A\hat{\chi}(\mathbf{u} - \mathbf{u}_o)$ and we use the fact that the Fourier transform of $\nabla^2 G(\mathbf{x})$ is $-4\pi^2 u^2 \hat{G}(\mathbf{u})$ to transform Eqn. (1.24). A tiny bit of algrebra then yields

$$\hat{G}(\mathbf{u}) = \frac{1}{\lambda^2 |\mathbf{u}|^2 - 1} \tag{1.31}$$

where $\lambda = 2\pi/k$.

The scattered field under the Born approximation is proportional to the Fourier transform of the object susceptibilty as filtered by the Fourier transform of the Green function. Due to the pole in the Green function, the scattered field only samples the susceptibilty on the sphere $|\mathbf{u}| = 1/\lambda$ in Fourier space. Since this sphere is the locus of endpoints of the plane wave propagation vectors, it is called the "wave normal surface."

While the spatial spectrum of the scattered field lies on the wave normal surface, it samples the the spatial spectrum of the object susceptibility on the shifted sphere $|\mathbf{u} - \mathbf{u_o}| = 1/\lambda$. This sphere is illustrated in the Fourier space of the object in Fig. 1.4. Since $\mathbf{u_o}$ is the wave normal for a plane wave satisfying Maxwell equations, it too lies on the wave normal sphere. This means that the origin of the Fourier space of the object, corresponding to $\mathbf{u} = \mathbf{u_o}$ is always represented in the scattered field. Other points in the Fourier space of the object represented in the scattered field depend on the orientation of $\mathbf{u_o}$.

While scattering from a single illumination wave samples only a shell of the object Fourier space, it is possible to sample a full band volume of the Fourier space by observing a series of scattered fields while rotating $\mathbf{u_o}$ about the origin. This is illustrated in Fig. 1.4 by the second sphere centered on $\mathbf{u_1}$. $\mathbf{u_1}$ represents the wave vector of a second illumination plane wave. Estimation of a 3D object from measurements collected over such an illumination series is called "diffraction tomography" and is the subject of chapter **??**. One may also fill in a 3D band volume by observing the scattering for a diversity of illumination wavelengths, which changes the radius of the sphere in Fig. 1.4. This approach is adopted in "optical coherence tomography," which is discussed in section 4.3.

Of course, one may choose to use more complex illumination than a simple plane wave. In this case, the Fourier spectrum of the scattered field is proportional to the covolution of the spatial spectra of the illumination and the object susceptibility projected on the wave

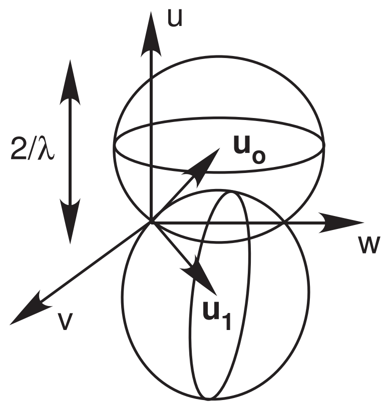

Figure 1.4. Fourier space of the object probed by scattered plane wave illumination.

normal sphere. Strategies that may take advantage of illumination coding are discussed in section 9.2.

It is also important to remember that the Born approximation is an approximation. Strong scattering and multipath scattering do not satisfy the Born approximation. One may attempt attempt to model stronger scattering by keeping more terms in the Born expansion, but convergence is slow under strong scattering conditions. One may alternatively solve Eqn. (1.23) by other strategies, using for example the coupled wave approach discussed in section 3.4.

The "Rytov approximation" interprets the Born scattering result from a different perspective. Where the Born approach assumes a solution to the inhomogeneous wave equation consisting of the superposition of the incident wave and the scattered wave, Rytov assumes a solution of the form

$$\psi(\mathbf{r}) = \psi_o(\mathbf{r})e^{\phi()}, \tag{1.32}$$

where $\phi(\mathbf{r})$ is a complex phase. Substituting this trial solution in Eqn. (1.23) yields

$$2\nabla\psi_o \cdot \nabla\phi + \psi_o\nabla^2\phi + \psi_o\left|\nabla\phi\right|^2 = k^2\chi(\mathbf{r})\psi_o(\mathbf{r}) \tag{1.33}$$

where we use $\left(\nabla^2 + k^2\right)\psi_o = 0$ to eliminate terms. Noting that

$$\begin{aligned} \nabla^2\left(\psi_o\phi\right) = &\quad \phi\nabla^2\psi_o + 2\nabla\psi_o \cdot \nabla\phi + \psi_o\nabla^2\phi \\ = &\quad -k^2\psi_o\phi + 2\nabla\psi_o \cdot \nabla\phi + \psi_o\nabla^2\phi \end{aligned} \tag{1.34}$$

Eqn. (1.33) becomes

$$\left(\nabla^2 + k^2\right)\left(\psi_o\phi\right) = \left(k^2\chi(\mathbf{r}) - \left|\nabla\phi(\mathbf{r})\right|^2\right)\psi_o(\mathbf{r}) \tag{1.35}$$

This equation may also be transformed into an integral equation using the Green function, yielding in this case

$$\phi(\mathbf{r}) = -\frac{1}{4\pi\psi_o(\mathbf{r})}\int\int\int \left(k^2\chi(\mathbf{r}') - |\nabla\phi(\mathbf{r}')|^2\right)\psi_o(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d\mathbf{r}' \tag{1.36}$$

The Rytov approximation consists of the assumption that $k^2\chi \gg |\nabla\phi(\mathbf{r}')|^2$, in which case Eqn. (1.37) reduces to

$$\phi(\mathbf{r}) = \frac{\psi_s(\mathbf{r})}{\psi_o(\mathbf{r})} \tag{1.37}$$

where ψ_s is the scattered field under the Born approximation as described in Eqn. (1.30). The scattered field under the Rytov approximation is the difference between the illumination field and the total field, e.g.

$$\psi_{sRytov} = \psi_o(\mathbf{r})\left[e^{\phi(\mathbf{r})} - 1\right] \tag{1.38}$$

The potential advantage of the Rytov approximation is that, while the Born approximation assumes that the total scattered field is much weaker than the illumination field, the Rytov approximation assumes only that the phase change of the field relative to the illumination field is small over the range $\lambda/2\pi\chi$. This assumption is likely to be valid for fields scattered over relatively small angles, but may be less valid for strong scattering over wide angles. Thus, one may choose the Born approximation for locally strong but spatially compact scatterers and the Rytov approximation for locally weak scatterers distributed over a larger volume.

While the Rytov and Born approaches intepret the scattered field differently, both ultimately estimate the Born integral (Eqn. (1.30)) from the scattered field and face the identical challenge of inverting this transformation to estimate the object suceptibility.

1.4 DIFFRACTION

Holography is a mechanism for capturing the complex field in a plane. Once one has determined the field in the measurement plane, one often propagates it back to the object space to form an image. The process of propagation may take an analog form in reconstructing a physical hologram or it may be implemented in digital processing. In either case, we need to understand the process of propagating a known field from one plane to another. This process is called *diffraction*.