Photon noise in holography

By Daniel Marks, Oct 28 2009

The Poisson "popcorn" Process

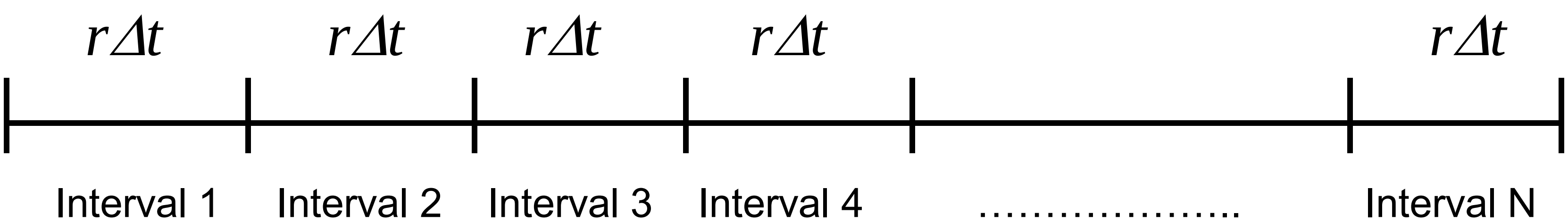
Photons arriving at a detector have similar statistics to popcorn popping.

At each time instant, there is an independent probability of a kernel popping.

The Poisson Process

For a small time interval Δt, there is a probability $r\Delta t$ of a kernel popping.

Over N intervals, the probability of P pops is:



$$\frac{N!}{P!(N-P)!}(r\Delta t)^P(1-r\Delta t)^{N-P}$$ Binomial distribution

The binomial limit

Total time is $T = N\Delta t$

Binomial distribution $$\frac{N!}{P!(N-P)!}\left(\frac{rT}{N}\right)^{P}\left(1-\frac{rT}{N}\right)^{N-P}$$

As N approaches infinity

$$\left(1-\frac{rT}{N}\right)^{N-P} = 1-(N-P)\left(\frac{rT}{N}\right)+\frac{(N-P)(N-P-1)}{2}\left(\frac{rT}{N}\right)^{2}+\ldots$$

$$\approx 1-rT+\frac{1}{2!}(rT)^{2}+\ldots = e^{-rT}$$

The binomial limit

$$\frac{N!}{P!(N-P)!}\left(\frac{rT}{N}\right)^P = \frac{N(N-1)(N-2)\ldots}{P!(N-P)(N-P-1)\ldots}\left(\frac{rT}{N}\right)^P \approx \frac{N^P}{P!}\left(\frac{rT}{N}\right)^P$$

Poisson distribution $\frac{(rT)^P}{P!}\exp(-rT)$

rT is average number of pops over time interval T

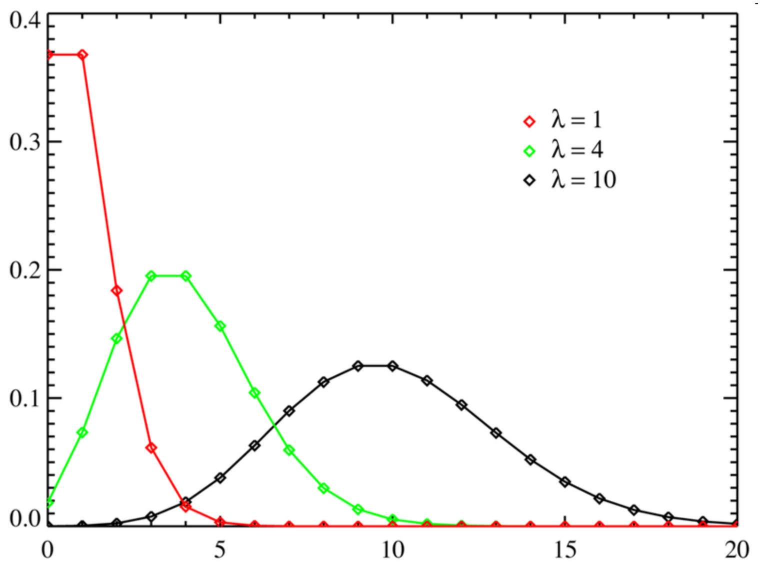


Figure shamelessly lifted from Wikipedia

Most important facts we use about Poisson distribution

As the average number of events rT gets large, Poisson approaches a Gaussian distribution.

rT is mean of distribution

rT is also the variance of the distribution!

Signal to noise ratio of Poisson process = $\dfrac{\text{mean}}{\sqrt{\text{variance}}}$

$$= \sqrt{rT}$$

How do we analyze the photon noise in optical systems?

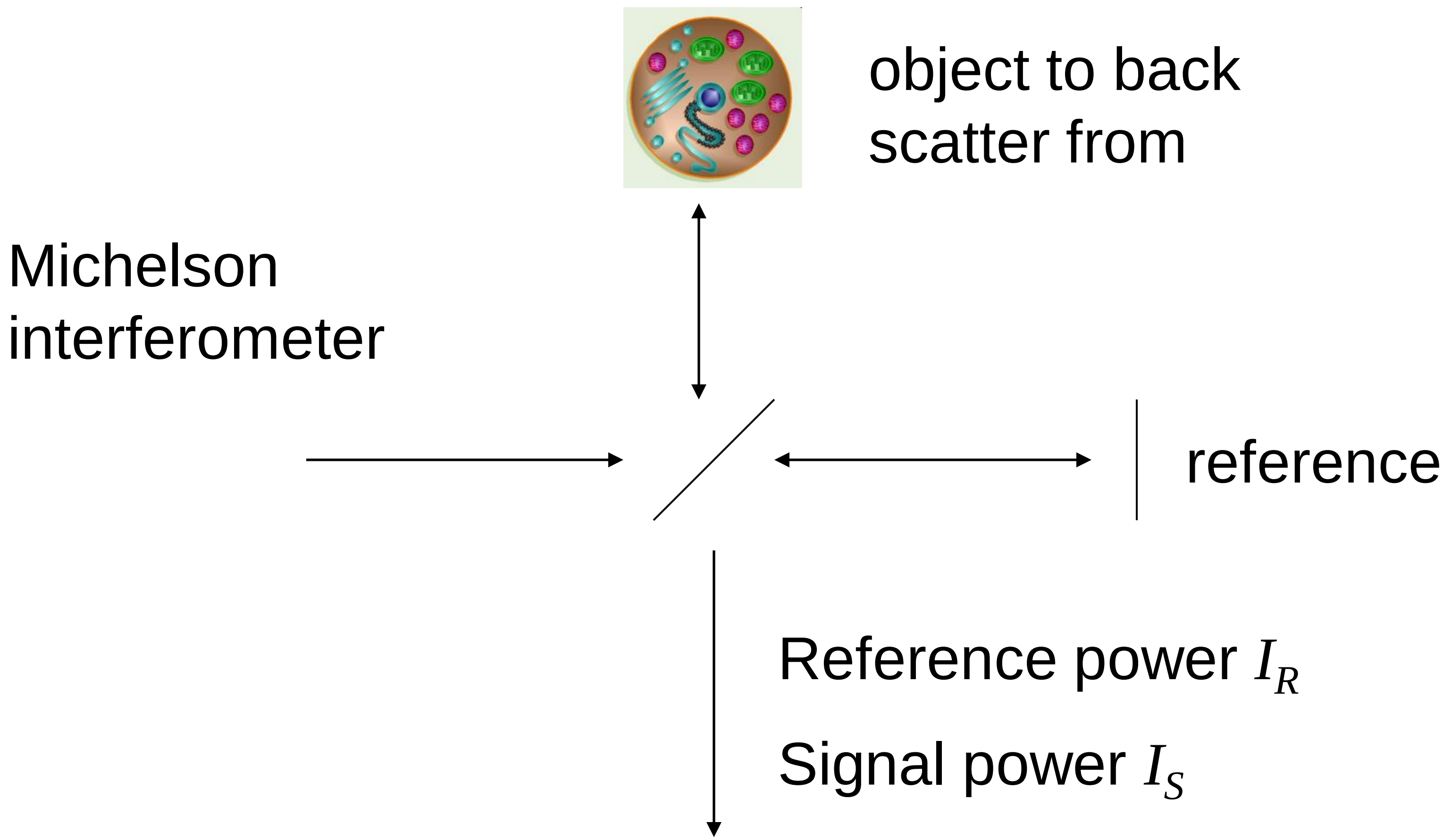
Important rule of thumb for quantum processes:

PHOTON NOISE OCCURS AT **DETECTION,**
NOT AT THE SOURCE.

We don’t know how many photons are emitted,
only how many we receive.

We start at the detector and work **backwards**
to find the mean/variance of unknown quantities.

A simple example, one interferometric measurement.



The interferometric advantage

$$I = I_R + I_S + 2\sqrt{I_R I_S}\cos\theta$$

I_R is constant

I_S changes

For $I_R^{h\nu} >> I_S$

$$I \approx I_R + 2\sqrt{I_R I_S}\cos\theta$$

and $I_R >> 2\sqrt{I_R I_S}\cos\theta$

The interferometric advantage continued

Number of signal photons $$\frac{2\sqrt{I_R I_S \cos^2\theta}}{h\nu} A\Delta t$$

A is area of detector, Δt is integration time,
$h\nu$ is photon energy

Number of reference photons $$\frac{I_R}{h\nu} A\Delta t$$

Variance in number of detected photons

$$\left[I_R + \frac{2\sqrt{I_R I_S}\cos\theta}{h\nu} \right] A\Delta t \approx \frac{I_R}{h\nu} A\Delta t$$

SNR of interferometric detection

$$\frac{\text{Signal photons}}{\sqrt{\text{Photon noise variance}}} = \frac{\dfrac{2\sqrt{I_R I_S \cos^2\theta}}{h\nu} A\Delta t}{\sqrt{\dfrac{I_R}{h\nu} A\Delta t}}$$

$$= \quad 2\sqrt{\frac{I_S \cos^2\theta}{h\nu} A\Delta t} \qquad \langle \cos^2\theta \rangle = \frac{1}{2}$$

$$= \sqrt{\frac{2I_S}{h\nu} A\Delta t}$$

… but this is the +/- the number of signal photons, independent of reference power.

The interferometric advantage

SNR achieves photon noise limit.

This can be achieved without
photon counting detectors! (e.g. photomultiplier)

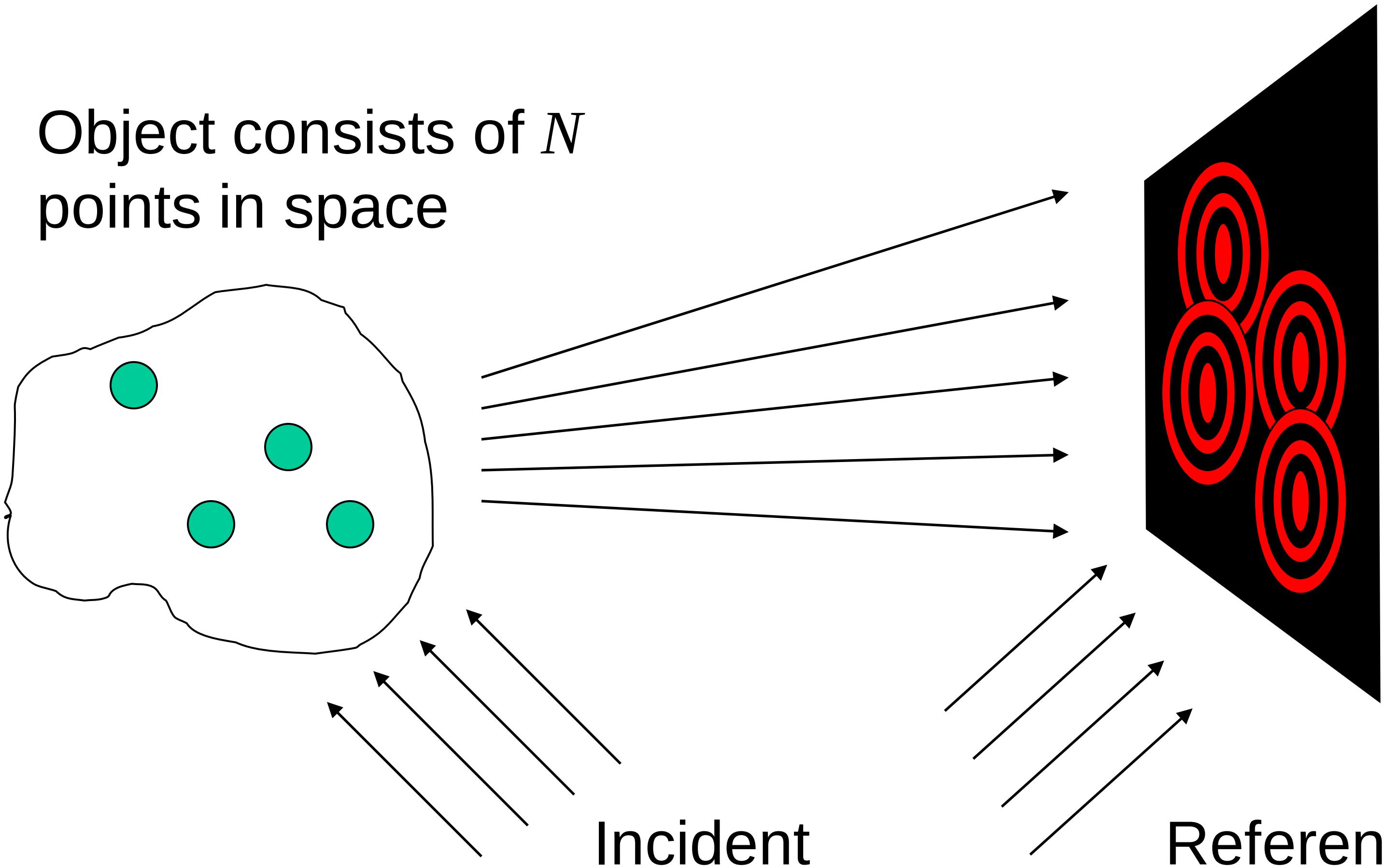
This is what enables holography, optical coherence tomography, etc. to use conventional detectors.

Reference power can be adjusted so thermal noise becomes small compared to photon noise.

Holography and photon noise

An abstract model of holography…

Object consists of N points in space

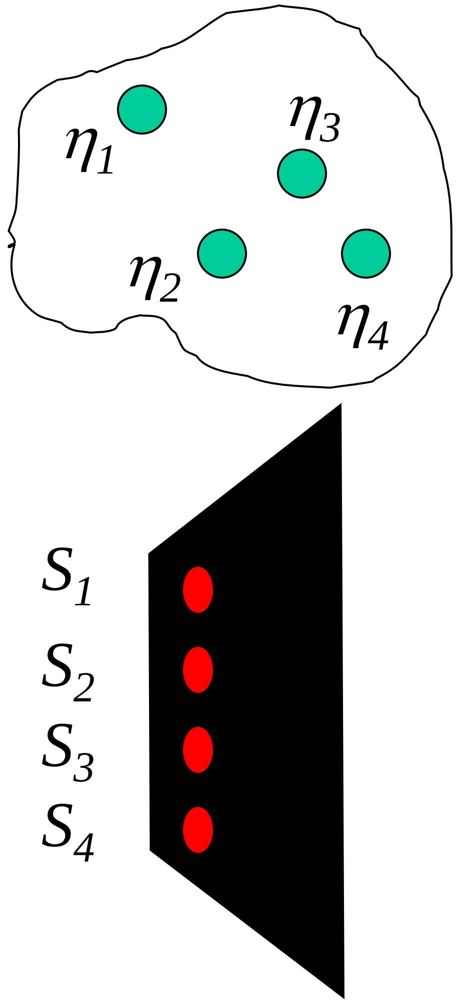


Interference pattern

Incident wavefront, amplitude E_0

Reference field E_R

Definitions of variables



Object consists of N points in space

The scattering amplitudes of these points are η_i to form a vector $\boldsymbol{\eta}$.

Likewise, the detected fields are a vector $\boldsymbol{S}$ with elements S_j

The optical system

The optical system relates the scattering amplitudes η_i to the detected fields S_j.

The optical system is modeled by a matrix H_{ij} such that

$$S_j = E_0 \sum_{i=1}^{N} \eta_i H_{ij}$$

Or in vector notation

$$\mathbf{S} = E_0 \mathbf{H} \boldsymbol{\eta}$$

Photon noise of the detected field S_j

Photon noise is primarily due to the reference beam

$$I_j = \frac{\eta}{2}\left|E_R + S_j\right|^2 \approx \frac{\eta}{2} E_R{}^2 + 2\,\mathrm{Re}\left\{\eta E_R S_j{}^*\right\}$$

η is impedance of free space

$$I_R = \frac{1}{2}\eta E_R^2$$

Average # of photons on detector j

$$\langle p \rangle = Var\ p = \frac{I_R A\Delta t}{h\nu} = \frac{A\Delta t}{2h\nu}\eta E_R^2$$

average and variance number of photons (Poisson process)

$$Var\ I_j = \left(\frac{h\nu}{A\Delta t}\right)^2 Var\ p = \frac{I_R h\nu}{A\Delta t} = \frac{h\nu}{2A\Delta t}\eta E_R^2$$

$$Var\ S_j = \left(\frac{1}{\eta E_R}\right)^2 Var\ I_j = \frac{h\nu}{2\eta A\Delta t}$$

Independent of reference power

Finding the covariance of the potential $\boldsymbol{\eta}$

$$Cov\ \mathbf{S} = E[\mathbf{S}\mathbf{S}^H] = \frac{h\nu}{2\eta A\Delta t}\mathbf{I}$$

$$\mathbf{S} = E_0\mathbf{H}\boldsymbol{\eta} \longrightarrow E_0^{-1}\mathbf{H}^{-1}\mathbf{S} = \boldsymbol{\eta}$$

$$Cov\ \boldsymbol{\eta} = E[\boldsymbol{\eta}\boldsymbol{\eta}^\dagger] = E_0^{-2}E[\mathbf{H}^{-1}\mathbf{S}\mathbf{S}^\dagger(\mathbf{H}^{-1})^\dagger]$$

$$Cov\ \boldsymbol{\eta} = \frac{h\nu}{2\eta E_0^2 A\Delta t}E[\mathbf{H}^{-1}(\mathbf{H}^{-1})^\dagger]$$

$$I_0 = \frac{1}{2}\eta E_0^2$$

$$Cov\ \boldsymbol{\eta} = \frac{h\nu}{4I_0 A\Delta t}\mathbf{H}^{-1}(\mathbf{H}^{-1})^\dagger$$

result

What if $\mathbf{H}$ is unitary?

Unitary $\mathbf{H}$ means…

$$\mathbf{H}\mathbf{H}^{\dagger} = \mathbf{H}^{\dagger}\mathbf{H} = \mathbf{I}$$

$$\mathbf{H}^{-1} = \mathbf{H}^{\dagger}$$

Examples of unitary transformations (up to a constant): Fraunhofer (far-field) diffraction, full-rank Fresnel diffraction matrix, identity matrix

$$Cov\ \boldsymbol{\eta} = \frac{hv}{4I_0 A\Delta t}\mathbf{H}^{-1}(\mathbf{H}^{-1})^{\dagger} = \frac{hv}{4I_0 A\Delta t}C^2\mathrm{I}$$

C is proportionality constant in unitary operator

…therefore for unitary transformations photon noise is uncorrelated at the scatterer, and dependent only on the total intensity incident on the scatterer

Fresnel diffraction

$\mathbf{r}_i$ are locations of scatterers regularly spaced

$\mathbf{r}'_j$ are locations of detected field regularly spaced

$$H_{ij} = -\frac{ik}{z}\exp\left[\frac{-ik}{z}\left|\mathbf{r}_i - \mathbf{r}'_j\right|^2\right]$$

$$H_{ij} = -\frac{ik}{z}\exp\left[\frac{-ik}{z}\left|\mathbf{r}_i\right|^2\right]\exp\left[\frac{-ik}{z}\left|\mathbf{r}'_j\right|^2\right]\exp\left[\frac{-2ik}{z}\mathbf{r}_i \cdot \mathbf{r}'_j\right]$$

discrete Fourier transform matrix