$$\vec{K} = k_0(\vec{s}-\vec{s}_0) \longrightarrow \vec{K} = (k_x, k_y)$$

presented by Devaney [Dev82]. First consider the inverse Fourier transform of the object function,

$$o(\vec{r}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} O(\vec{K})e^{j\vec{K}\cdot\vec{r}}\, d\vec{K}. \quad \text{// } dk_x dk_y \text{ or } d\alpha d\beta \tag{157}$$

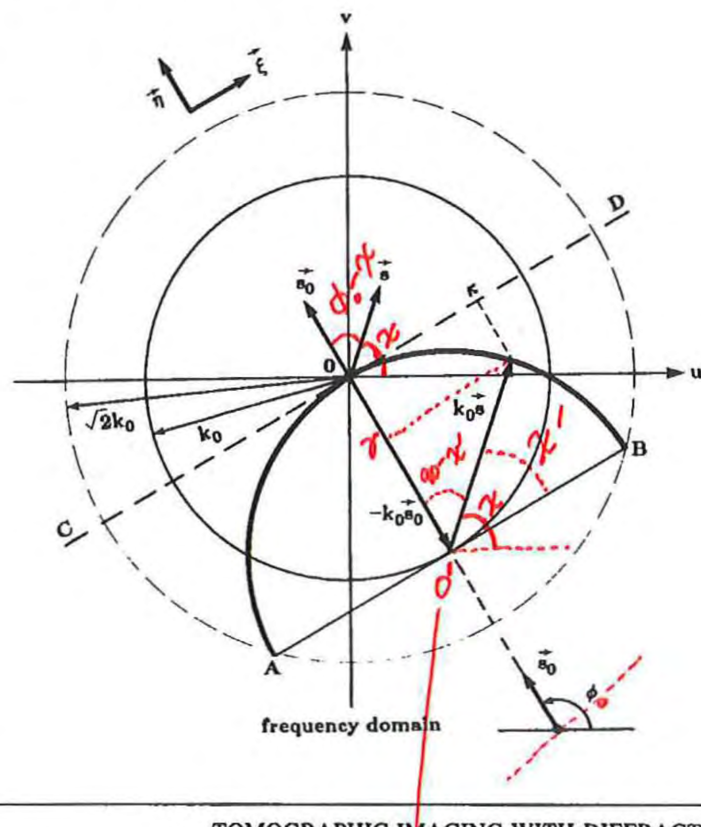
This integral most commonly represents the object function in terms of its Fourier transform in a rectangular coordinate system representing the frequency domain. As we have already discussed, a diffraction tomography experiment measures the Fourier transform of the object along circular arcs; thus it will be easier to perform the integration if we modify it slightly to use the projection data more naturally. We will use two coordinate transformations to do this: the first one will exchange the rectangular grid for a set of semicircular arcs and the second will map the arcs into their plane wave decomposition.

We first exchange the rectangular grid for semicircular arcs. To do this we represent $\vec{K} = (k_x, k_y)$ in (157) by the vector sum

$$\vec{K} = k_0(\vec{s}-\vec{s}_0) \tag{158}$$

where $\vec{s}_0 = (\cos\phi_0, \sin\phi_0)$ and $\vec{s} = (\cos\chi, \sin\chi)$ are unit vectors representing the direction of the wave vector for the transmitted and the received plane waves, respectively. This coordinate transformation is illustrated in Fig. 6.18.

Fig. 6.18: *The $k_0\vec{r}_0$ and $k_0\vec{s}$ used in the backpropagation algorithm are shown here. (From [Pan83].)*



$\cos(90-\chi') = \frac{(k_0\vec{s})\cdot(k_0\vec{s}_0)}{k_0^2} = (\vec{s}\cdot\vec{s}_0)$

$\cos(90-\chi') = \frac{\gamma}{k_0}$

$\sqrt{1-(\vec{s}\cdot\vec{s}_0)^2} = \sqrt{1-\frac{\gamma^2}{k_0^2}} = \frac{|K|}{k_0}$

In this new coordinate system. k_0 is added to value $\gamma - k_0 \rightarrow \gamma$

Jacobian

$$\begin{vmatrix} \frac{\partial k_x}{\partial \chi} & \frac{\partial k_x}{\partial \phi_0} \\ \frac{\partial k_y}{\partial \chi} & \frac{\partial k_y}{\partial \phi_0} \end{vmatrix} = \begin{vmatrix} -k_0 \sin\chi & k_0 \sin\phi_0 \\ k_0 \cos\chi & -k_0\cos\phi_0 \end{vmatrix}$$

$$= k_0^2 (\sin\chi\cos\phi_0 - \cos\chi\sin\phi_0)$$
$$= k_0^2 (\sin(\chi-\phi_0))$$
$$= k_0^2 \sqrt{1-\cos^2(\chi-\phi_0)}$$
$$= k_0^2 \sqrt{1-\cos^2(\phi_0-\chi)}$$

To find the Jacobian of this transformation write

$$k_x = k_0 (\cos\chi - \cos\phi_0) \tag{159}$$

$$k_y = k_0 (\sin\chi - \sin\phi_0) \tag{160}$$

and

$$dk_x dk_y = |k_0^2 \sin(\chi-\phi_0)| \, d\chi \, d\phi_0 \tag{161}$$

$$= k_0 \sqrt{1-\cos^2(\chi-\phi_0)} \, d\chi \, d\phi_0 \tag{162}$$

$$= k_0 \sqrt{1-(\vec{s}\cdot\vec{s}_0)^2} \, d\chi \, d\phi_0 \tag{163}$$

and then (157) becomes

$$o(\vec{r}) = \frac{1}{(2\pi)^2} \left(\frac{1}{2}\right) k_0^2$$

for all angles, double coverage is generated.

$$\cdot \int_0^{2\pi} \int_0^{2\pi} \sqrt{1-(\vec{s}\cdot\vec{s}_0)^2} \, O[k_o(\vec{s}-\vec{s}_0)] \, e^{jk_0(\vec{s}-\vec{s}_0)\cdot\vec{r}} \, d\chi \, d\phi_0. \tag{164}$$

The factor of 1/2 is necessary because as discussed in Section 6.4.1 the (χ, ϕ_0) coordinate system gives a double coverage of the (k_x, k_y) space.

$$\vec{S}_0 \cdot \vec{S} = |\vec{S}||\vec{S}_0| \cos(\phi_0-\chi) = \cos(\phi_0-\chi)$$

This integral gives an expression for the scattered field as a function of the (χ, ϕ_0) coordinate system. The data that are collected will actually be a function of ϕ_0, the projection angle, and κ, the one-dimensional frequency of the scattered field along the receiver line. To make the final coordinate transformation we take the angle χ' to be relative to the (κ, γ) coordinate system. This is a more natural representation since the data available in a diffraction tomography experiment lie on a semicircle and therefore the data are available only for $0 \leq \chi' \leq \pi$. We can rewrite the χ' integral in (164) by noting

(new origin: O') $\phi_0 = \pi/2$

$$\cos\chi' = \kappa/k_0 \tag{165}$$

$$\sin\chi' = \gamma/k_0 \tag{166}$$

and therefore

$$-\sin\chi' d\chi' = dk/k_0$$

$$d\chi' = \frac{-1}{\cancel{k_0}\gamma} d\kappa. \tag{167}$$

$$d\chi' = -\frac{k_0}{\gamma} \cdot \frac{dk}{k_0} = -\frac{dk}{\gamma}$$

* alternative?

$$\sin\chi' = (k_0-\gamma)/k_0$$
$$-\frac{(k_0-\gamma)}{k_0} d\chi' = dk/k_0$$
$$d\chi' = -\frac{k_0}{k_0-\gamma} \frac{dk}{k_0} = -\frac{dk}{k_0-\gamma}$$

The χ integral becomes $\quad -k_0 \leq k \leq k_0$

$$-\frac{1}{k_0} \int_{-k_0}^{k_0} \frac{d\kappa}{\gamma} |\kappa| O[k_0(\vec{s}-\vec{s}_0)] e^{jk(\vec{s}-\vec{s}_0)\cdot\vec{r}} \, d\kappa. \tag{168}$$

$$(104) \quad U_B(\alpha, l_0) = \frac{j}{2\sqrt{k_0^2-\alpha^2}} e^{j\sqrt{k_0^2-\alpha^2}\, l_0}\, O(\alpha, \sqrt{k_0^2-\alpha^2} - k_0)$$

Using the Fourier Diffraction Theorem as represented by (104) we can approximate the Fourier transform of the object function, O, by a simple function of the first-order Born field, u_B, at the receiver line. Thus the object function in (168) can be written

$\cancel{\phi_0 = 90^\circ}$

$$O[k_0(\vec{s}-\vec{s}_0)] = -2\gamma j U_B(\kappa, \gamma - k_0)e^{-j\gamma l_0}. \tag{169}$$

In addition, if a rotated coordinate system is used for $\vec{r} = (\xi, \eta)$ where

$\phi_0 = \frac{\pi}{2}$ $\quad \phi_0 = \frac{\pi}{2} + \phi$

$$\xi = x \sin\phi - y\cos\phi \tag{170}$$

and

$\phi \rightarrow \phi'$

$$\eta = x\cos\phi + \sin\phi, \tag{171}$$

$$\begin{bmatrix} \cos(\frac{\pi}{2}+\phi) & +\sin(\frac{\pi}{2}+\phi) \\ -\sin(\frac{\pi}{2}+\phi) & \cos(\frac{\pi}{2}+\phi) \end{bmatrix} = [\sin$$

then the dot product $k_0(\vec{s} - \vec{s}_0)\cdot\vec{r}$ can be written

In rotated coordinates, $\phi' = 90$

$k_0(\vec{s}-\vec{s}_0)\cdot\vec{r}$

$$\kappa\xi + (\gamma - k_0)\eta. \tag{172}$$

$= (k_0(\cos\chi' - \cos\phi'), k_0(\sin\chi' - \sin\phi'))\cdot(\xi, \eta)$

$= (k_0\cos\chi', k_0\sin\chi' - k_0)\cdot(\xi, \eta)$

The coordinates (ξ, η) are illustrated in Fig. 6.19. Using the results above we can now write the χ integral of (164) as

$$\frac{2j}{k_0}\int_{-k_0}^{k_0} d\kappa|\kappa|U_B(\kappa, \gamma - k_0)e^{-j\gamma l_0}e^{\kappa\xi + (\gamma - k_0)\eta} \tag{173}$$

$= (\kappa, \gamma - k_0)\cdot(\xi, \eta)$

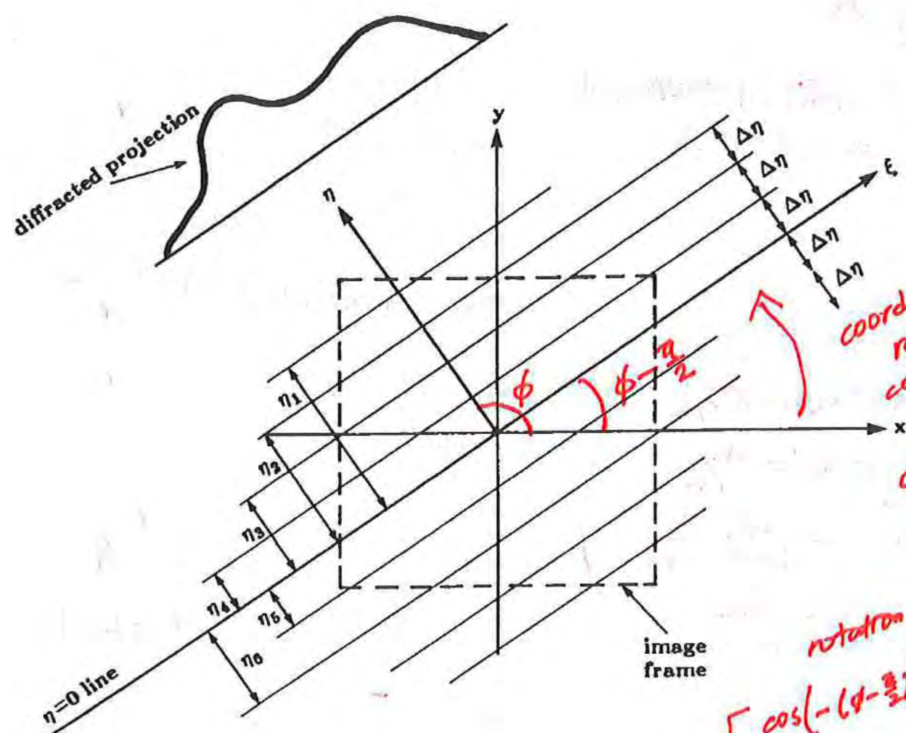


Fig. 6.19: *In backpropagation the projection is backprojected with a depth-dependent filter function. At each depth, η, the filter corresponds to propagating the field a distance of $\Delta\eta$. (From [Sla83].)*

coordinate rotation counter-clockwise

//

object or function rotation clockwise

rotation transformation:

$$\begin{bmatrix} \cos(-(\phi-\frac{\pi}{2})) & -\sin(-(\phi-\frac{\pi}{2})) \\ \sin(-(\phi-\frac{\pi}{2})) & \cos(-(\phi-\frac{\pi}{2})) \end{bmatrix} = \begin{bmatrix} \sin\phi & -\cos\phi \\ \cos\phi & \sin\phi \end{bmatrix}$$

frequency variable

(frequency filter)

and the equation for the object function in (164) becomes

$$o(\vec{r})=\frac{jk_0}{(2\pi)^2}\int_0^{2\pi} d\phi_0 \int_{-k_0}^{k_0} d\kappa\,|\kappa|\, U_B(\kappa,\ \gamma-k_0)e^{-j\gamma l_0}e^{j\kappa\xi+j(\gamma-k_0)\eta}. \tag{174}$$

To bring out the filtered backpropagation implementation, we write here separately the inner integration:

$$\Pi_\phi(\xi,\ \eta)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma_\phi(\omega)H(\omega)G_\eta(\omega)\exp\,(j\omega\xi)\,d\omega \tag{175}$$

where

$$H(\omega)=|\omega|, \qquad |\omega|\le k_0, \tag{176}$$

$$=0, \qquad |\omega|>k_0 \tag{177}$$

$$G_\eta(\omega)=\exp\,[j(\sqrt{k_0^2-\omega^2}-k_0)\eta], \qquad |\omega|\le k_0, \tag{178}$$

$$=0, \qquad |\omega|>k \tag{179}$$

and

$$\Gamma_\phi(\omega)=U_B(\kappa,\ \gamma-k_0)e^{-j\gamma l_0}. \tag{180}$$

phase delay factor. for each "plane"

Without the extra filter function $G_\eta(\omega)$, the rest of (175) would correspond to the filtering operation of the projection data in x-ray tomography. The filtering as called for by the transfer function $G_\eta(\omega)$ is depth dependent due to the parameter η, which is equal to $x \cos \phi + y \sin \phi$.

In terms of the filtered projections $\Pi_\phi(\xi,\ \eta)$ in (175), the reconstruction integral of (174) may be expressed as

$$f(x,\ y)=\frac{1}{2\pi}\int_0^{2\pi} d\phi\,\Pi_\phi(x\sin\phi-y\cos\phi,\ x\cos\phi+y\sin\phi). \tag{181}$$

The computational procedure for reconstructing an image on the basis of (175) and (181) may be presented in the form of the following steps:

Step 1: In accordance with (175), filter each projection with a separate filter for each depth in the image frame. For example, if we chose only nine depths as shown in Fig. 6.19, we would need to apply nine different filters to the diffracted projection shown there. (In most cases for a 128 × 128 reconstruction grid, the number of discrete depths chosen for filtering the projection will also be around 128. If there are much less than 128, spatial resolution will suffer.)

Step 2: To each pixel $(x,\ y)$ in the image frame, in accordance with (181), allocate a value of the filtered projection that corresponds to the